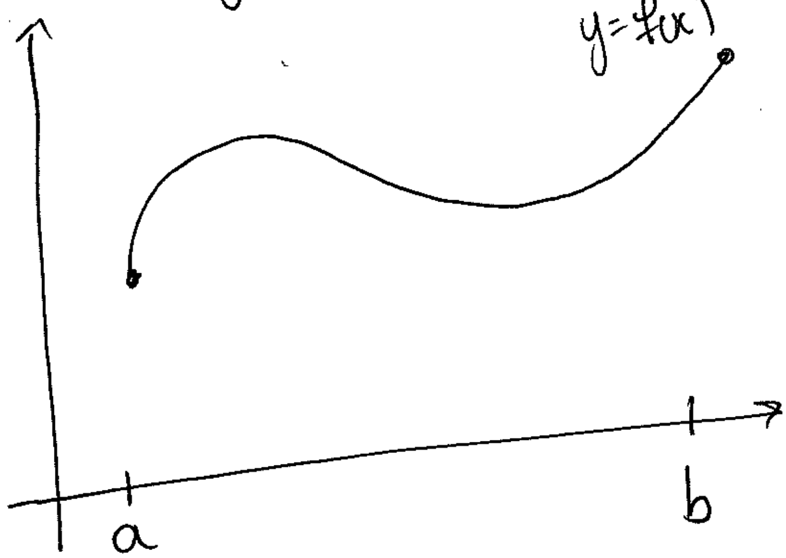
March 3, 2014

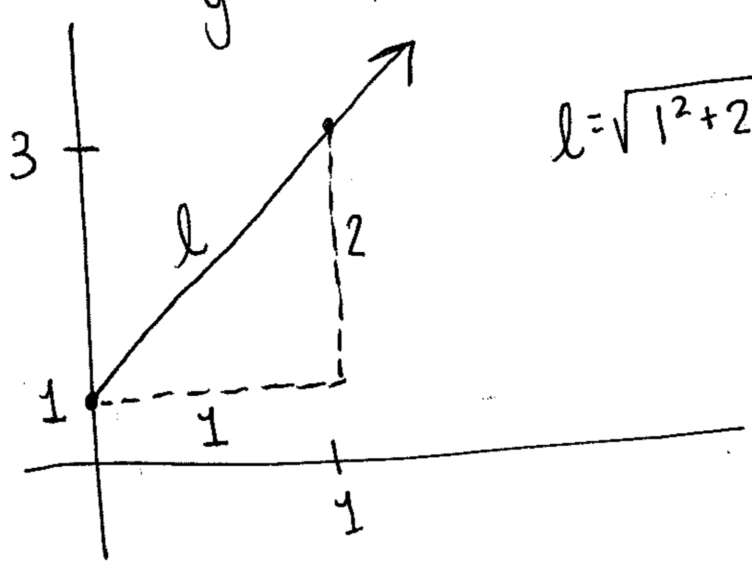
Arc Length

Want to find the length of a curve:



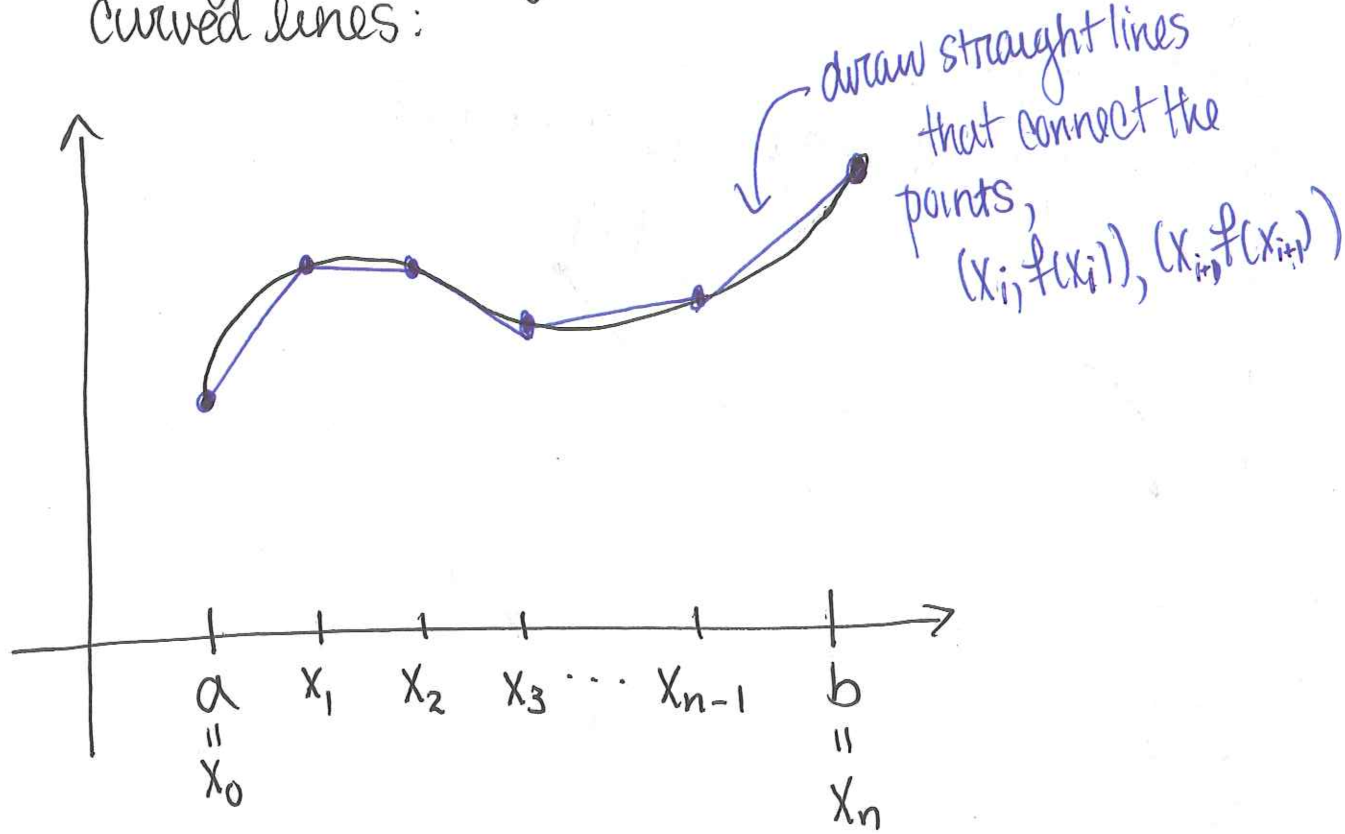
If $f(x)$ is a straight line, this is fairly easy:

$y=2x+1$, what is arc length from 0 to 1?

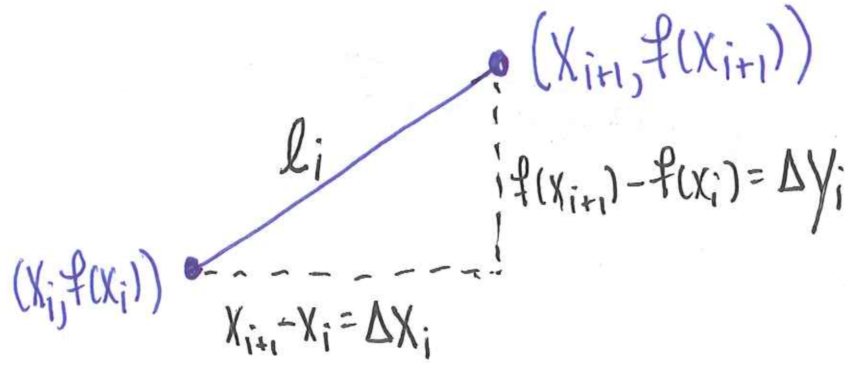


$$l=\sqrt{1^2+2^2}=\sqrt{5}$$

Idea, use what we know about arc length of straight lines to find the arc length of curved lines:



What is the length of one of these lines?



So $l_i = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)} = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$

Remember $\Delta x \sim dx$, $\Delta y \sim dy$

arc length: add up the length of all line segments

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x_i \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

So $\frac{dy}{dx} = f'(x)$.

Arc length Formula:

$$\int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Examples:

(1) arc length of $f(x)=2x+1$ from 0 to 1.

$f'(x)=2$

Arc length $= \int_0^1 \sqrt{1+(2)^2} = \int_0^1 \sqrt{5}\, dx = \sqrt{5}\, x \Big|_0^1 = \underline{\underline{\sqrt{5}}}$

(2) Find arc length of $f(x)=x^{3/2}$ from 1 to 2.

$$f'(x)=\frac{3}{2}x^{1/2}$$

$$\int_1^2 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_1^2 \sqrt{1+\frac{9}{4}x}\,dx$$

Need substitution: $u=1+\frac{9}{4}x$

$du=\frac{9}{4}dx$

$$\int_{1+\frac{9}{4}\cdot 1}^{1+\frac{9}{4}\cdot 2} \frac{4}{9}\sqrt{u}\,dx = \frac{4}{9}\cdot\frac{2}{3}\cdot u^{3/2}\Big|_{13/4}^{22/4}$$

$$=\frac{8}{27}\left(\frac{22}{4}\right)^{3/2}-\frac{8}{27}\left(13/4\right)^{3/2}$$

(3) Find the arc length of $f(x)=x^2-\frac{1}{8}\ln x$ from 1 to 2

$$f'(x)=2x-\frac{1}{8x}$$

$$\int_1^2 \sqrt{1+\left(2x-\frac{1}{8x}\right)^2}\,dx = \int_1^2 \sqrt{1+4x^2-\frac{4x}{8x}+\frac{1}{64x^2}}\,dx$$

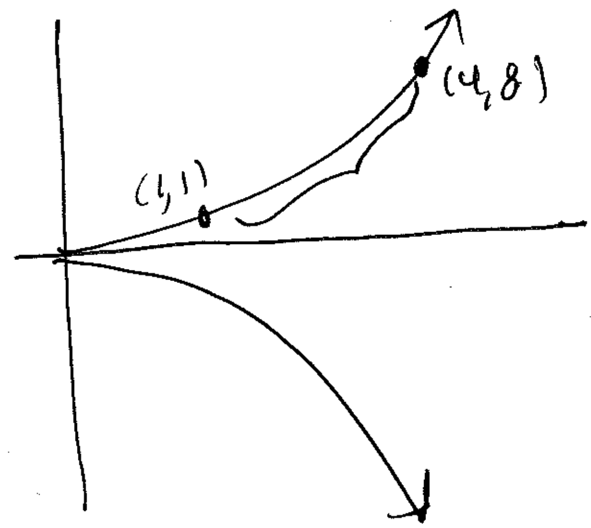
$$=\int_1^2 \sqrt{1+4x^2-\frac{1}{2}+\frac{1}{64x^2}}\,dx$$

$$=\int_1^2 \sqrt{4x^2+\frac{1}{2}+\frac{1}{64x^2}}\,dx = \int_1^2 \sqrt{\left(2x+\frac{1}{8x}\right)^2}\,dx$$

$$=\int_1^2 2x+\frac{1}{8x}\,dx = x^2+\frac{1}{8}\ln x\Big|_1^2 = 4+\frac{1}{8}\ln 2-1 = \underline{3+\frac{1}{8}\ln 2}$$

(4) Find the length of the arc of $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$

$$y^2 = x^3 \Rightarrow y = \pm x^{3/2}$$ ← want to only consider positive sign since we want arc length of positive points



$$f'(x) = \frac{3}{2}x^{1/2}$$

$$\int_1^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_1^4$$

$$\approx 7.634$$